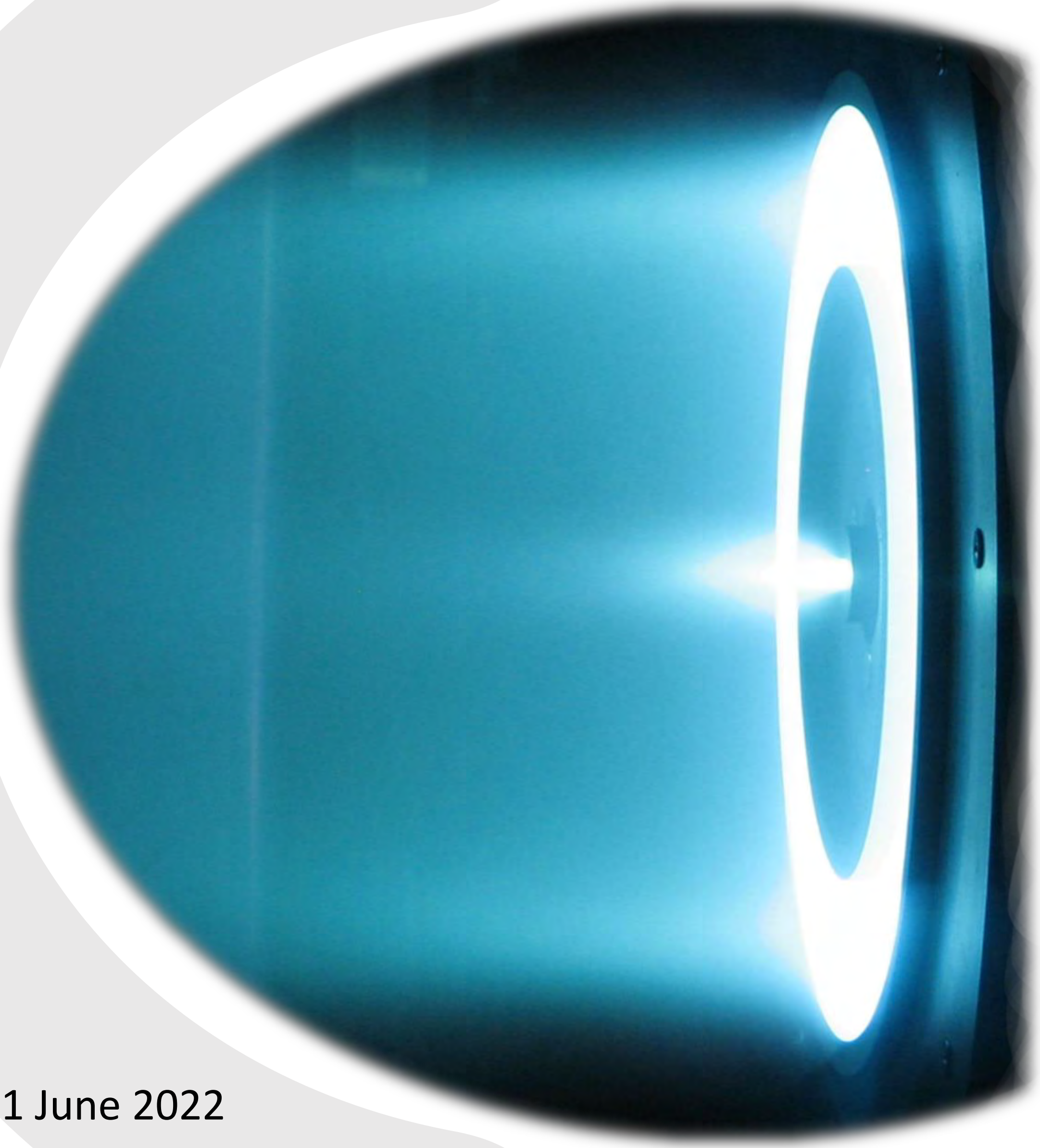

PLASMA INSTABILITY ANALYSIS IN HALL THRUSTERS

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SEM 8 PROJECT

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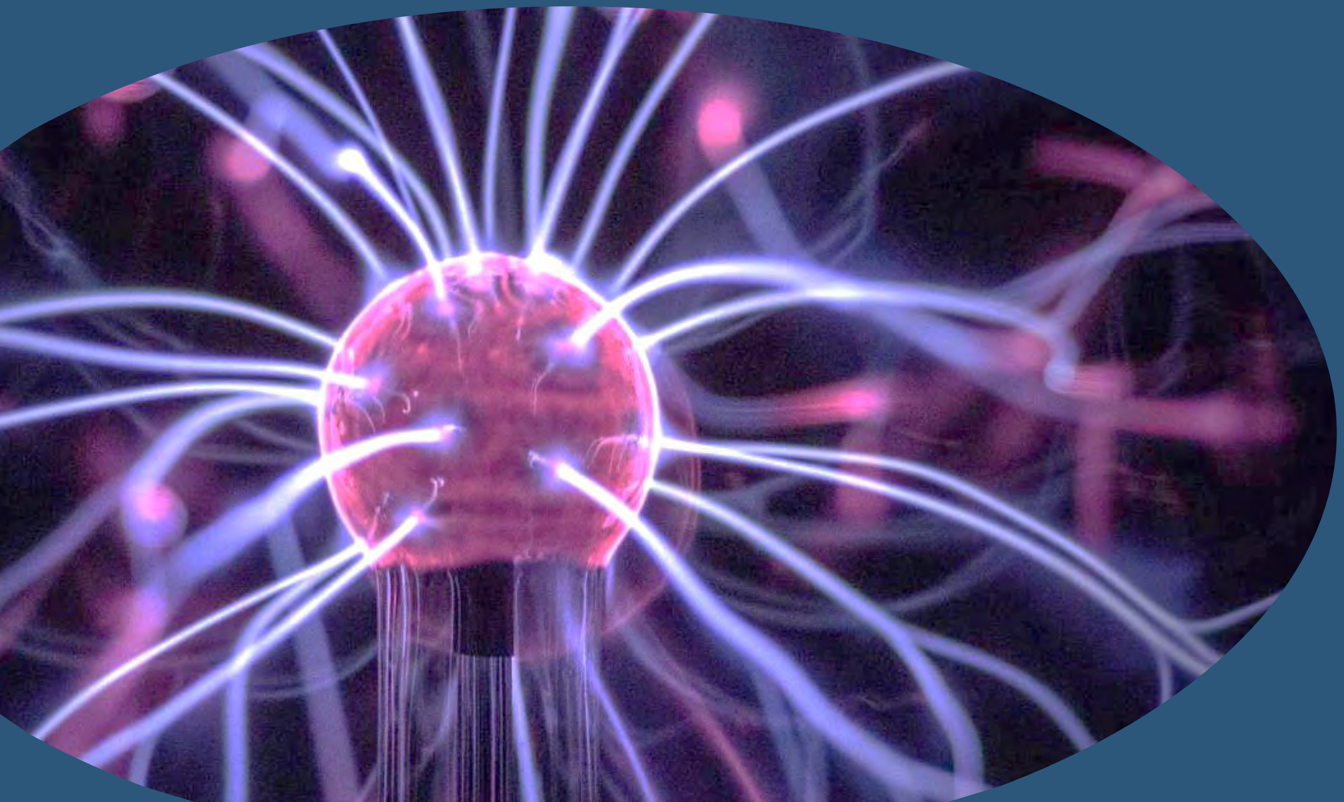

Nurturing talent...promoting excellence

मौलिक विज्ञान प्रकर्ष केन्द्र

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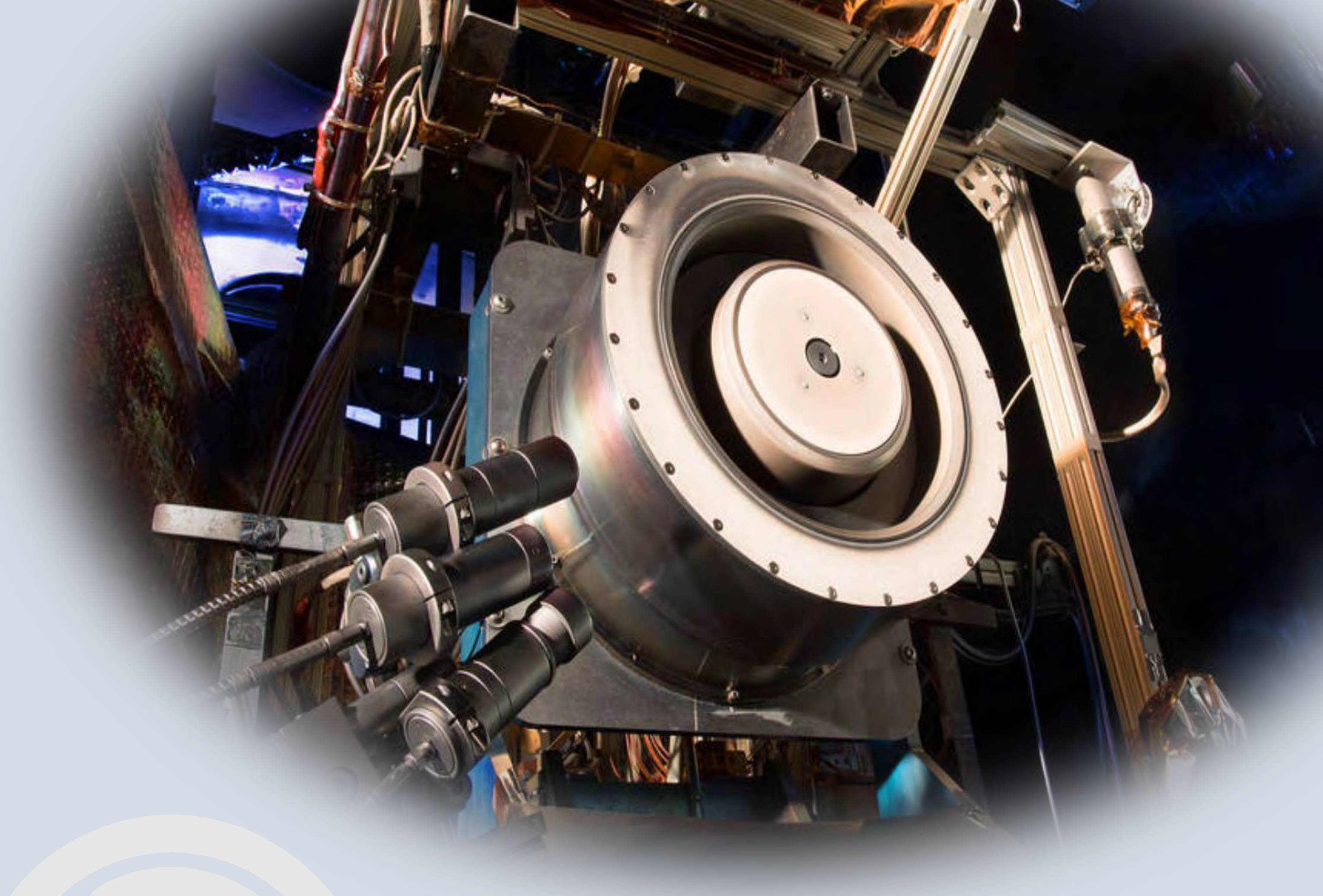
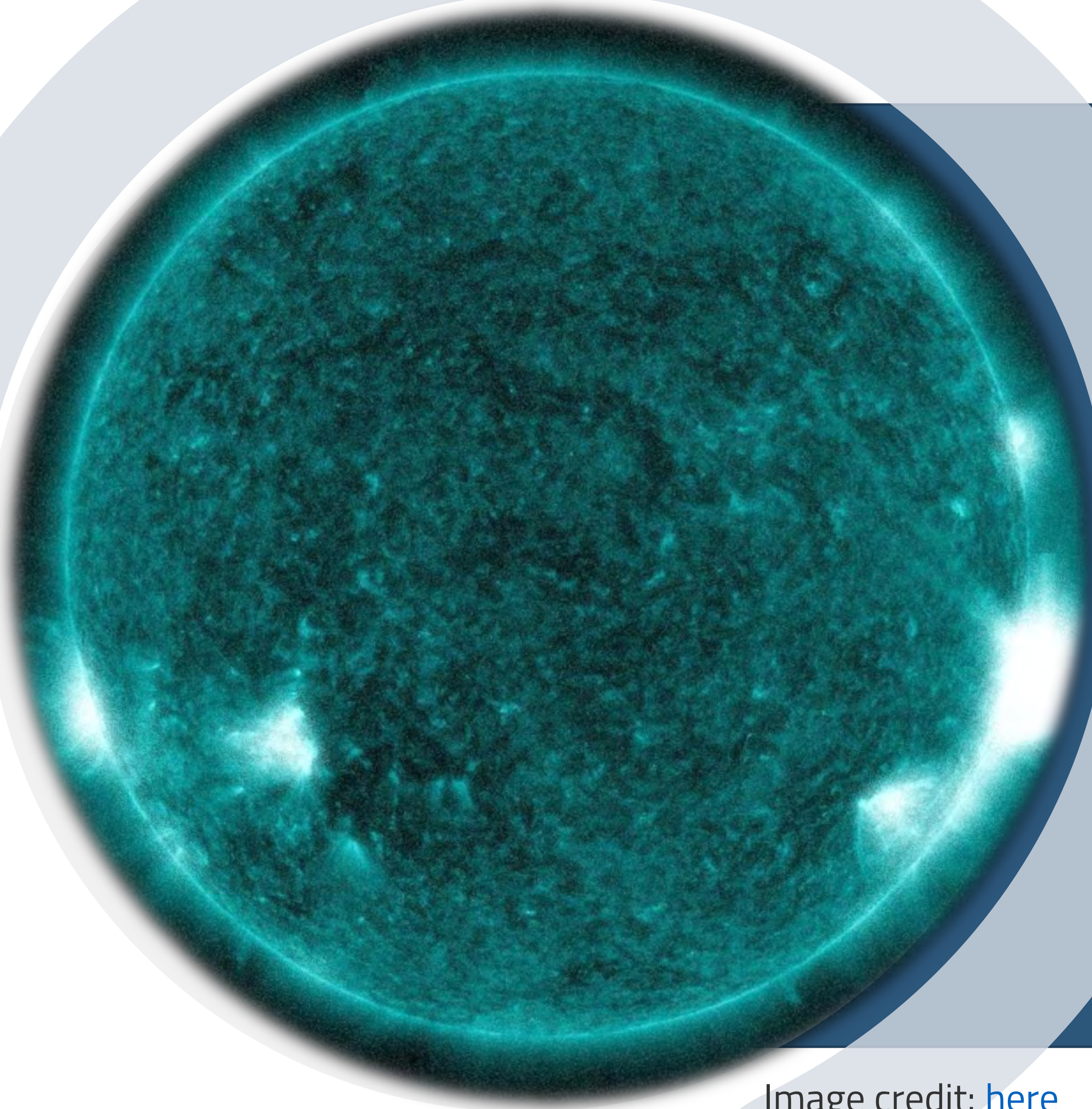


Image credit: SEP Hall Thruster, NASA

SESSION 1

INTRODUCTION



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WHAT IS A PLASMA ?

DEFINITION

4th state of matter which is a quasi-neutral medium of unbound positive and negative charged particles.

Image credit: here

FUNDAMENTAL PARAMETERS

- Particle Density
- Temperature
- Steady State Magnetic Field

VLASOV - MAXWELL EQUATIONS

Time evolution of the distribution function of plasma consisting of charged particles with long-range interaction, e.g. Coulomb.

Collisionless Vlasov - Maxwell equation,

$$\frac{\partial f_\sigma}{\partial t} + \nabla \cdot \mathbf{v}_\sigma f_\sigma - q_\sigma \frac{\partial}{\partial \mathbf{p}} \cdot \left(\mathbf{E} + \frac{\mathbf{v}_\sigma}{c} \times \mathbf{B}\right) f_\sigma = 0$$

Anatoly Vlasov, Russian Theoretical Physicist

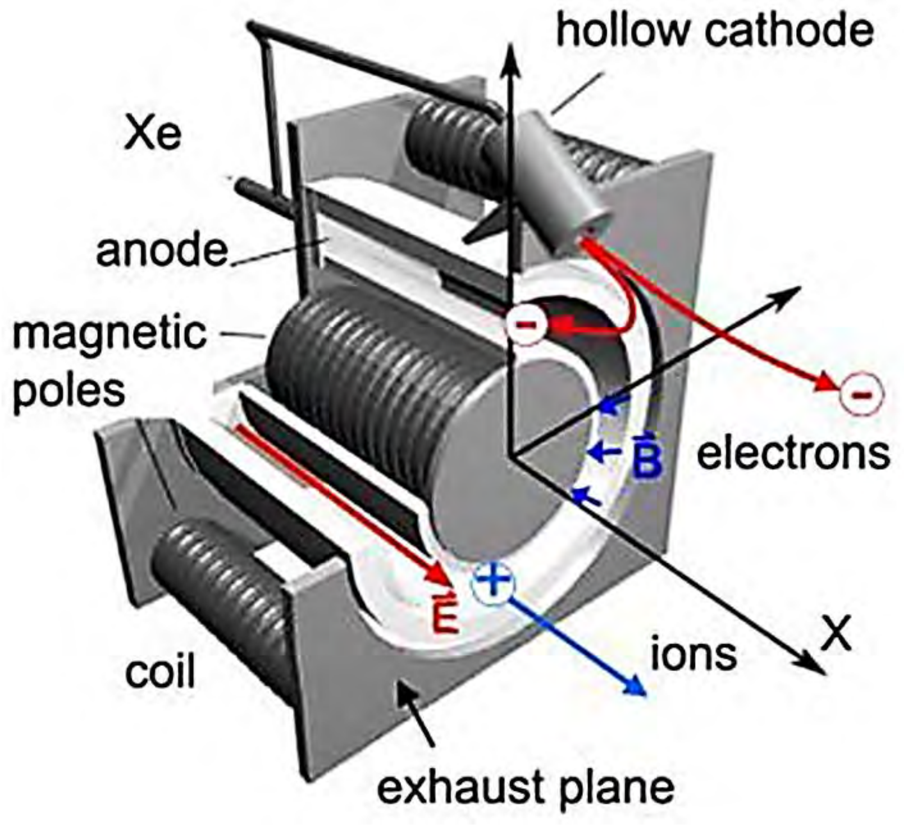
HALL THRUSTERS

PRINCIPLE

Gridless Ion Thrusters that makes use of **Hall effect**.

Permanent magnets provide the external magnetic barrier for electrons.

These confined electrons ionizes the neutral atoms that are fed in.



SCHEMATIC OF HALL THRUSTER

Image credit: L. Dubois, et. al.; 2018

THE HALL THRUSTER PROBLEM

MOTIVATION

The electron transport across the magnetic confinement (**ANOMALOUS TRANSPORT**) is a mystery.

Anything ranging from Electron collisions with the walls and secondary electron emission to instabilities and turbulence could be responsible for this anomalous transport along the magnetic field.

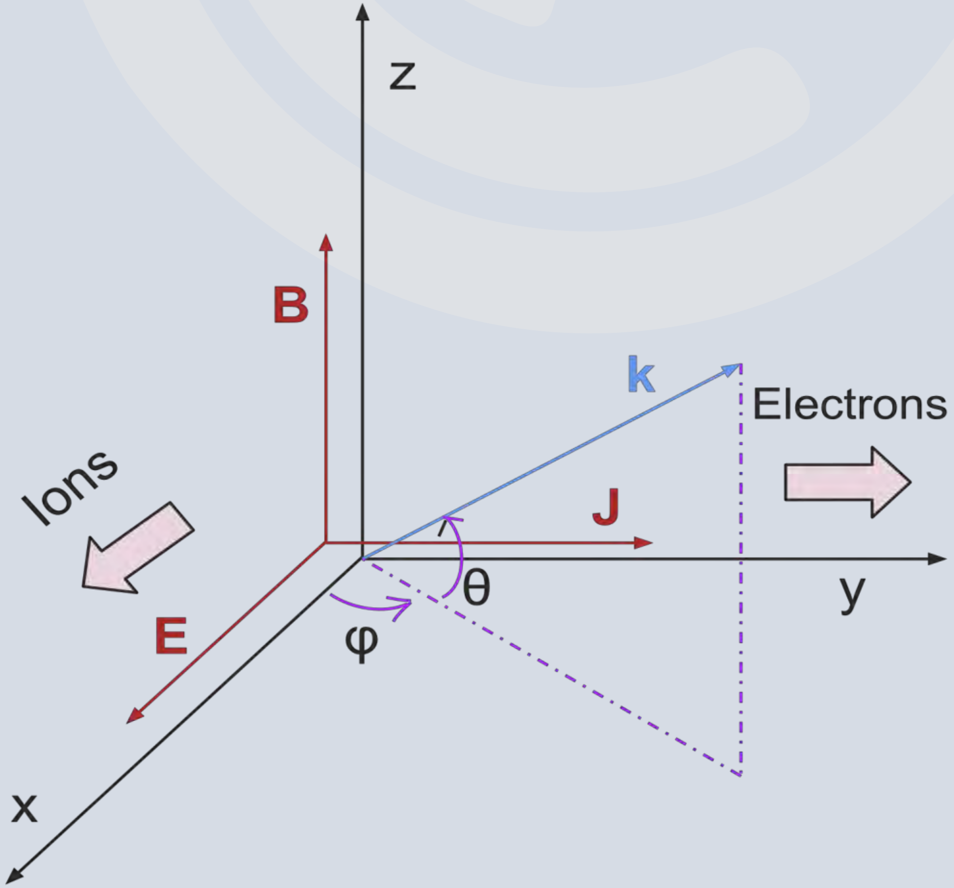
THE HALL THRUSTER PROBLEM

Linear Perturbation Model

Localized Cartesian Coordinates.

Length scale of B is such that the electrons undergo an $E_0 \times B_0$ drift while the ions being massive are unaffected and remain unmagnetized ($|\omega| \gg \omega_{ci}$).

This essentially leads to separation of charges along the two perpendicular directions.



SESSION 2

DISPERSION RELATION

OUTLINE OF DERIVATION

Perturbed Vlasov Equation:

Under the Electrostatic Perturbations,

$$\vec{B} = \boldsymbol{B} + 0$$
$$\vec{E} = \boldsymbol{E} - \nabla\phi_1$$
$$f_\sigma = f_{\sigma 0} + f_{\sigma 1}$$

Pertubation is of the form,

$$x_1 \sim \exp \mathrm{i}(\mathbf{k}\cdot\mathbf{x} - \omega t)$$

where $\boldsymbol{B}$ and $\boldsymbol{E}$ are initial uniform fields, the linearized Vlasov equation takes the form,

$$\frac{\partial f_{\sigma 1}}{\partial t} + \frac{\partial}{\partial \mathbf{x}}\cdot \mathbf{v} f_{\sigma 1} + \frac{q_\sigma}{m_\sigma}\frac{\partial}{\partial \mathbf{v}}\cdot(\mathbf{v}\times\mathbf{B} + \mathbf{E}) f_{\sigma 1} = \frac{q_\sigma}{m_\sigma}\nabla\phi_1 \cdot \frac{\partial f_{\sigma 0}}{\partial \mathbf{v}}$$

OUTLINE OF DERIVATION

Perturbed Vlasov Equation –
Method of Characteristics:

$$\frac{\partial f_{\sigma 1}}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} f_{\sigma 1} + \frac{q_\sigma}{m_\sigma} (\mathbf{v} \times \mathbf{B} + \mathbf{E}) \cdot \frac{\partial}{\partial \mathbf{v}} f_{\sigma 1} = \frac{q_\sigma}{m_\sigma} \nabla \phi_1 \cdot \frac{\partial f_{\sigma 0}}{\partial \mathbf{v}}$$

which, along a **unperturbed single particle phase-space trajectory** would look like,

$$\frac{\mathrm{d}}{\mathrm{d}t} f_{\sigma 1}(\mathbf{x}(t), \mathbf{v}(t), t) \Big|_{\mathrm{unperturbed}} = \frac{q_\sigma}{m_\sigma} \nabla \phi_1 \cdot \frac{\partial f_{\sigma 0}}{\partial \mathbf{v}}$$

OUTLINE OF DERIVATION

Perturbed Vlasov Equation –

Method of Characteristics:

Given the Perturbed Differential equation,

$$\frac{\mathrm{d}}{\mathrm{d}t} f_{\sigma 1}(\mathbf{x}(t), \mathbf{v}(t), t) \bigg|_{\text{unperturbed}} = \frac{q_\sigma}{m_\sigma} \nabla \phi_1 \cdot \frac{\partial f_{\sigma 0}}{\partial \mathbf{v}}$$

Formal Solution is,

$$f_{\sigma 1}(\mathbf{x}, \mathbf{v}, t) = \frac{q_\sigma}{m_\sigma} \int_{-\infty}^{t} \mathrm{d}t' \left[\nabla \phi_1 \cdot \frac{\partial f_{\sigma 0}}{\partial \mathbf{v}} \right]_{\mathbf{x}=\mathbf{x}(t'), \mathbf{v}=\mathbf{v}(t')}$$

where the <u>integration is over the unperturbed trajectory</u>.

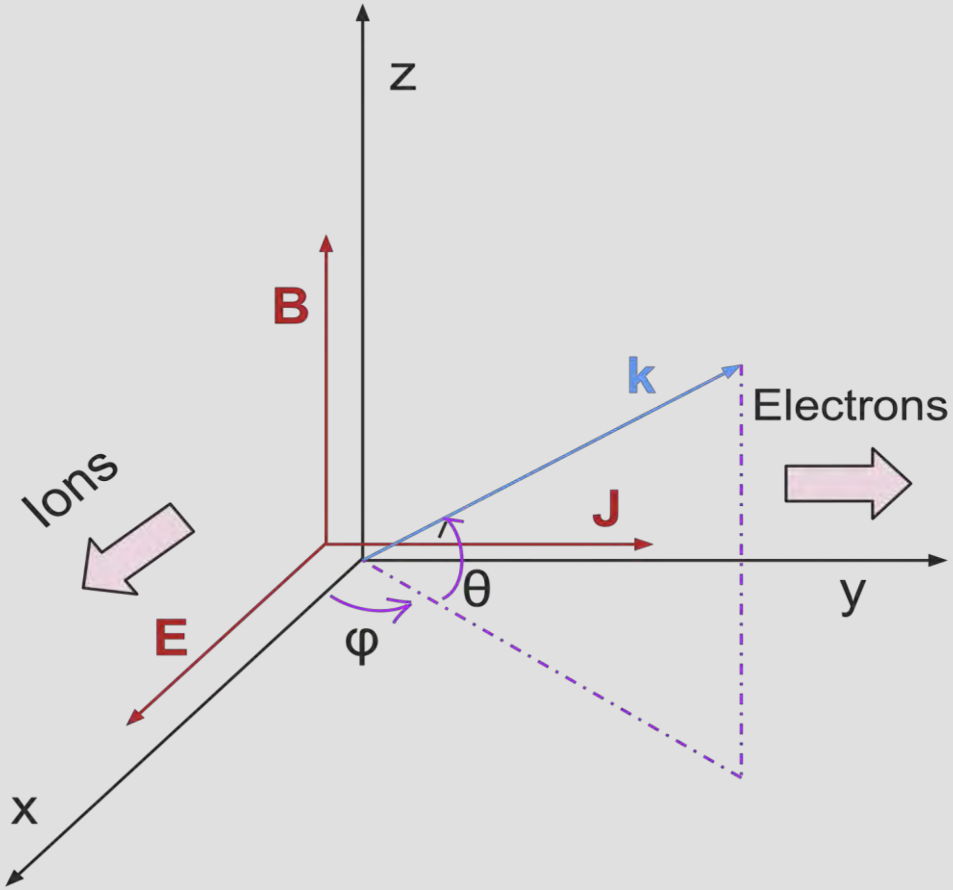
OUTLINE OF DERIVATION

Perturbed Vlasov Equation –
Method of Characteristics –
- **Single Particle Solution:**

We start with the Lorentz equation of Motion for a charged particle,

$$\dot{\mathbf{v}} = \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \frac{q}{m}\left(E\hat{x} + v_y B\hat{x} - v_x B\hat{y}\right)$$

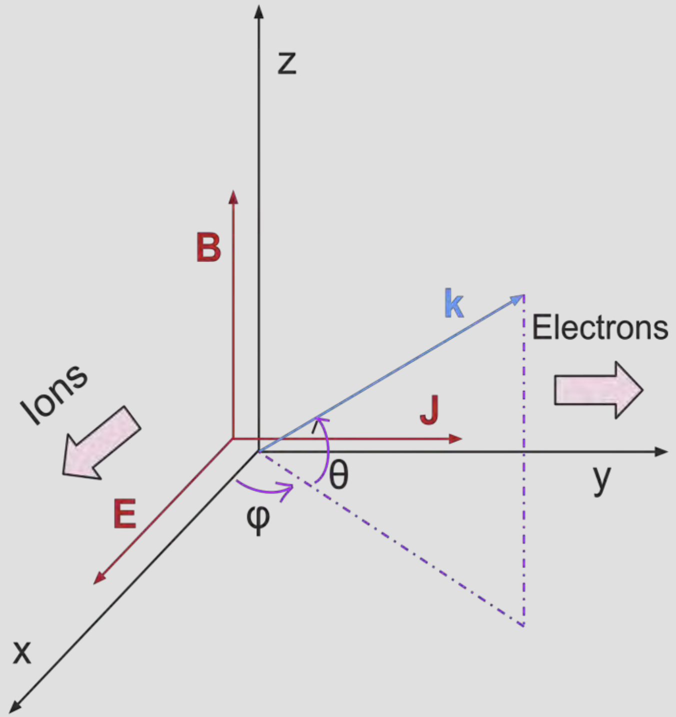
E and B are uniform in space



Schematics of the Hall Thruster Problem

OUTLINE OF DERIVATION

Perturbed Vlasov Equation –
Method of Characteristics –
- **Single Particle Solution:**



For the Boundary Condition,

$$\mathbf{v}(0) = \mathbf{v}_{\mathrm{x}} + v_0\hat{\mathbf{v}}_{\boldsymbol{y}} + v_{\parallel}\hat{\mathbf{v}}_{\boldsymbol{z}}$$

We have the unperturbed trajectory as,

$$\mathbf{v}(t') = v_{\parallel}\hat{\mathbf{v}}_{\mathbf{z}} + \cos[\omega_{c\sigma}(t'-t)]\hat{\mathbf{v}}_{\mathbf{x}} + (\sin[\omega_{c\sigma}(t'-t)] - \mathrm{v}_0)\hat{\mathbf{v}}_{\boldsymbol{y}}$$

$$\mathbf{x}(t') = \mathbf{x} + v_{\parallel}(t'-t)\hat{\mathbf{v}}_{\mathbf{z}} + \frac{1}{\omega_{c\sigma}}\begin{Bmatrix} \sin[\omega_{c\sigma}(t'-t)]\,\hat{\mathbf{v}}_{\mathbf{x}} \\ +(\cos[\omega_{c\sigma}(t'-t)] - 1)\hat{\mathbf{v}}_{\boldsymbol{y}} \end{Bmatrix} - \mathrm{v}_0(t'-t)\hat{\mathbf{v}}_{\boldsymbol{y}}$$

OUTLINE OF DERIVATION

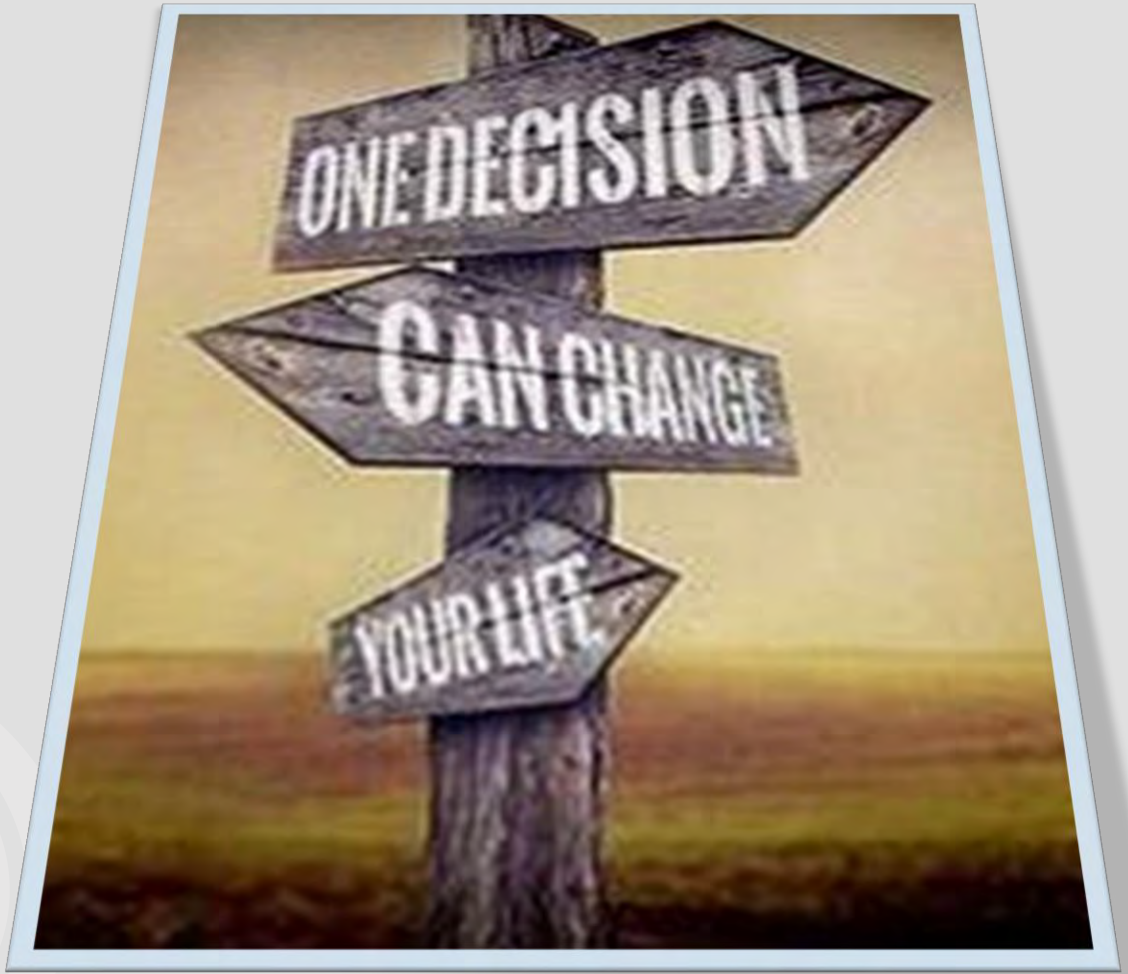
Perturbed Vlasov Equation –
Method of Characteristics –

- Single Particle Solution –
- **Maxwellian distribution:**

CHOOSE the normalized distribution function as,

$$f_{\sigma 0}(\mathbf{v}) = \frac{n_{\sigma 0}}{(2\pi v_{T\sigma}^2)^{3/2}} \exp - \left(\frac{v_z^2 + v_\perp'^2}{2\, v_{T\sigma}^2} \right)$$

where v_z^2 and $v_\perp'^2 = (\mathbf{v}_\perp - \mathbf{v}_0)^2$ are Constants of Motion.



OUTLINE OF DERIVATION

Perturbation in Distribution Function –

Formal Solution is,

$$f_{\sigma 1}(\mathbf{x}, \mathbf{v}, t) = \frac{q_\sigma}{m_\sigma} \int_{-\infty}^{t} \mathrm{d}t' \left[\nabla \phi_1 \cdot \frac{\partial f_{\sigma 0}}{\partial \mathbf{v}} \right]_{\mathbf{x}=\mathbf{x}(t'),\mathbf{v}=\mathbf{v}(t')}$$

On integration along unperturbed trajectory,

$$f_{\sigma 1}(\mathbf{x}, \mathbf{v}, t) = -\frac{q_\sigma}{m_\sigma\, v_{T\sigma}^2} \phi_1 f_{\sigma 0}(\mathbf{v}) \left[\begin{array}{c} 1 + (\omega - \mathbf{k} \cdot \mathbf{v}_0) \sum\limits_{m,\, n} J_m\left(\dfrac{k_\perp v_\perp'}{\omega_{c\sigma}} \right) \\ \times J_n\left(\dfrac{k_\perp v_\perp'}{\omega_{c\sigma}} \right) \dfrac{\mathrm{e}^{\mathrm{i}(n-m)\varphi}}{(k_z v_z + k_y v_0 - \omega + n\omega_{c\sigma})} \end{array} \right]$$

OUTLINE OF DERIVATION

Perturbation in Number Density –

The perturbed number density is,

$$n_{\sigma 1} = \int f_{\sigma_1}\, d\mathbf{v}$$

And so,

$$n_{\sigma 1} = -\frac{q_\sigma}{m_\sigma\, v_{T\sigma}^2}\phi_1(\mathbf{x},t)\, n_{\sigma 0}\left[\begin{array}{c} 1 + \dfrac{\omega - k_y v_0}{\sqrt{2}\, k_z v_{T\sigma}} \mathrm{e}^{-k_\perp^2 r_{L\sigma}^2} \\ \times \displaystyle\sum_n I_n(k_\perp^2 r_{L\sigma}^2) \int_{-\infty}^{\infty} \mathrm{d}\xi\, \frac{\mathrm{e}^{-\xi^2}}{\xi - \alpha_{n\sigma}} \end{array}\right]$$

DISPERSION RELATION

Further we use the Poisson's Equation,

$$\nabla^2 \phi_1 = 4\pi(q_i n_i + q_e n_e)$$

to get the **warm magnetized plasma** (Doppler shifted electrons) **electrostatic dispersion relation** ,

$$\frac{1}{T_e}\left[1 + \frac{k^2 \mathrm{v}_{Te}^2}{\omega_{pe}^2}\right] + \frac{1}{T_i}\left[1 + \frac{\omega}{\sqrt{2}\, k_z v_{T_i}} Z\left(\frac{\omega}{\sqrt{2} k_z \mathrm{v}_{T_i}}\right)\right]$$
$$+ \frac{(\omega - k_y v_0)}{\sqrt{2}\, k_z v_{T_e}} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-k_\perp^2 r_{L\sigma}^2}\, I_n(k_\perp^2 r_{L\sigma}^2)\, Z\left(\frac{\omega - k_y v_0 - n\omega_{ce}}{\sqrt{2}\, k_z v_{T_e}}\right) = 0$$

DISPERSION RELATION

Warm magnetized plasma (Doppler shifted electrons) **electrostatic dispersion relation** ,

$$\frac{1}{T_e}\left[1+\frac{k^2 \mathrm{v}_{Te}^2}{\omega_{pe}^2}\right]+\frac{1}{T_i}\left[1+\frac{\omega}{\sqrt{2}\,k_z v_{T_i}} Z\left(\frac{\omega}{\sqrt{2} k_z \mathrm{v}_{T_i}}\right)\right]$$
$$+\frac{(\omega-k_y v_0)}{\sqrt{2}\,k_z v_{T_e}}\sum_{n=-\infty}^{\infty}\mathrm{e}^{-k_\perp^2 r_{L\sigma}^2}\; I_n(k_\perp^2 r_{L\sigma}^2)\, Z\left(\frac{\omega-k_y v_0-n\omega_{ce}}{\sqrt{2}\,k_z v_{T_e}}\right)=0$$

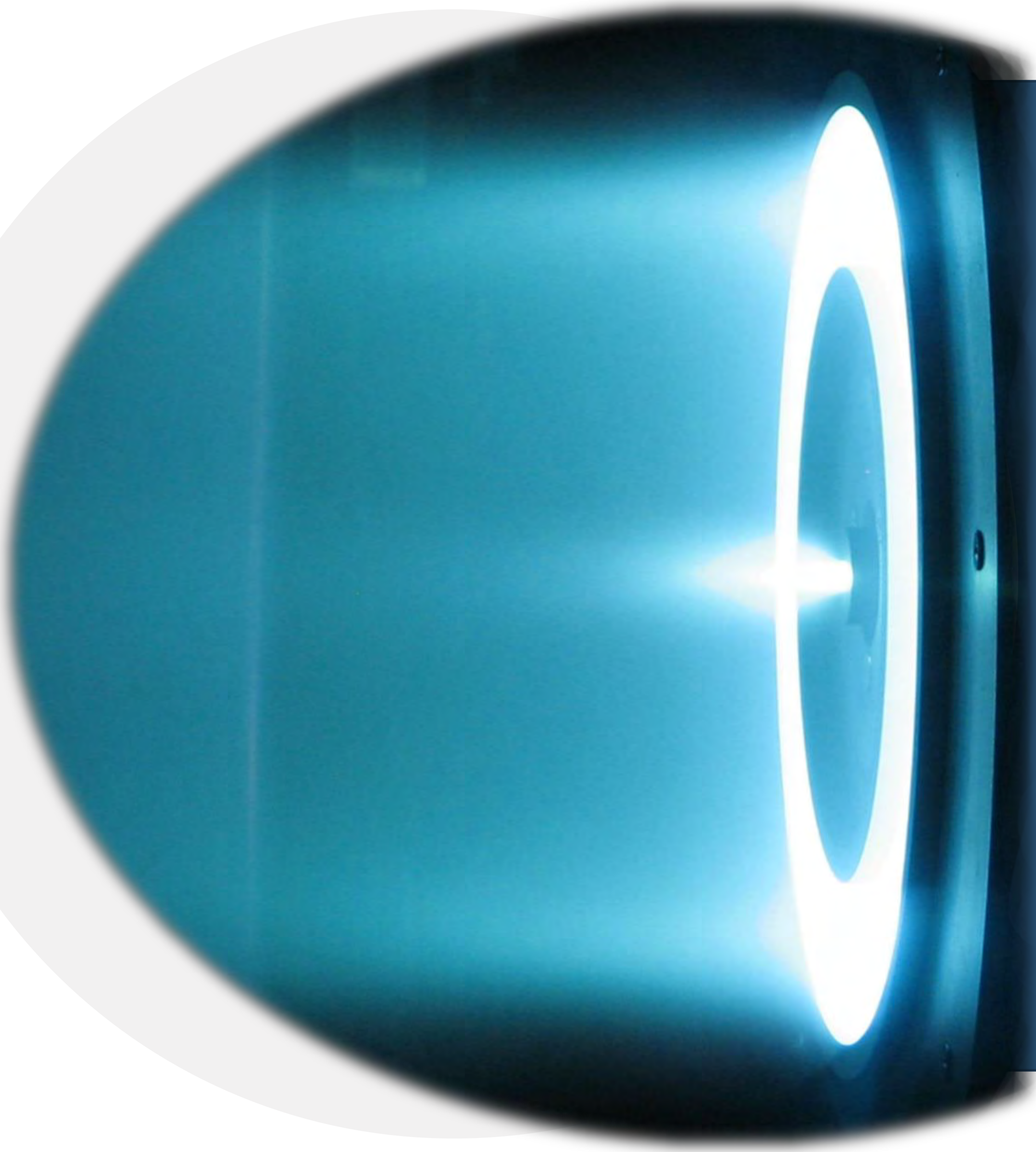
I_n is Modified Bessel's Function of first kind

Plasma Distribution Function,

$$Z(\alpha)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{d}\xi\,\frac{\exp(-\xi^2)}{(\xi-\alpha)}$$

SESSION 3

DISCUSSION



INSTABILITIES

Techniques for Instability Analysis by –

- ❑ Numerical stimulations
- ❑ Analytical means
  - Natural Frequencies of the system
  - Growth rates and Instabilities

ANALYTICAL INSTABILITY ANALYSIS

COLD PLASMA LIMIT – $\left|(\omega - k_y v_0 - n\omega_{c\sigma})/k_\parallel v_T\right| \gg 1$

We have the general dispersion relation,

$$\frac{1}{T_e}\left[1+\frac{k^2 \mathrm{v}_{Te}^2}{\omega_{pe}^2}\right]+\frac{1}{T_i}\left[1+\frac{\omega}{\sqrt{2}\,k_z v_{T_i}} Z\left(\frac{\omega}{\sqrt{2}k_z \mathrm{v}_{T_i}}\right)\right]$$
$$+\frac{(\omega - k_y v_0)}{\sqrt{2}\,k_z v_{T_e}} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-k_\perp^2 r_{L\sigma}^2}\; I_n(k_\perp^2 r_{L\sigma}^2)\, Z\left(\frac{\omega - k_y v_0 - n\omega_{ce}}{\sqrt{2}\,k_z v_{T_e}}\right) = 0$$

which in the cold plasma limit would be,

$$1 - \frac{k_z^2}{k^2}\frac{\omega_{pi}^2}{\omega^2} - \frac{\mathrm{e}^{-k_\perp^2 r_{Le}^2}}{k^2\lambda_{De}^2}\left[\frac{k_z^2 v_{T_e}^2\, I_0(k_\perp^2 r_{Le}^2)}{(\omega - k_y v_0)^2} - \sum_{n=1}^{\infty}\frac{2n_e^2\omega_{ce}^2}{(\omega - k_y v_0)^2 - n_e^2\omega_{ce}^2} I_n(k_\perp^2 r_{Le}^2)\right] = 0$$

ANALYTICAL INSTABILITY ANALYSIS

COLD PLASMA LIMIT

Modified Two Stream Instability –

For the case where $\omega - k_y v_0 \ll \omega_{ce}$ (ions are unmagnetized),

$$1 + \sum_{\sigma} \frac{e^{-k_\perp^2 r_{L\sigma}^2}}{k^2 \lambda_{D\sigma}^2} \left[\begin{array}{c} I_0(k_\perp^2 r_{L\sigma}^2)\big(1 + \alpha_{0\sigma} Z(\alpha_{0\sigma})\big) \\ + \sum_{n=1}^{\infty} I_n(k_\perp^2 r_{L\sigma}^2) \left[2 + \alpha_{0\sigma} \{Z(\alpha_{n\sigma}) + Z(\alpha_{-n\sigma})\}\right] \end{array} \right] = 0$$

$$k_y^2 \left(1 + \frac{\omega_{pe}^2}{\omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega^2}\right) + k_z^2 \left(1 - \frac{\omega_{pe}^2}{\left(\omega - k_y v_{e0}\right)^2} - \frac{\omega_{pi}^2}{\omega^2}\right)$$

$$- k_y^4 \left[\frac{3}{4\omega_{ce}^2} \frac{r_{Le}^4}{\lambda_{De}^2} + \frac{3\omega_{ci}^2}{\omega^4} \frac{r_{Li}^4}{\lambda_{Di}^2}\right] + k_y^2 k_z^2 \left[\frac{\omega_{pe}^2 r_{Le}^2}{\left(\omega - k_y v_{e0}\right)^2} + \frac{\omega_{pi}^2 r_{Li}^2}{\omega^2}\right] = 0$$

ANALYTICAL INSTABILITY ANALYSIS

COLD PLASMA LIMIT

Modified Two Stream Instability –

To 2nd order we have,

$$1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2 k_z^2}{(\omega - k_y v_0)^2 k^2} + \frac{\omega_{pe}^2 k_y^2}{\omega_{ce}^2 k^2} = 0$$

This may be thought of as coupling between $\omega^2 \sim \omega_{pi}^2$, and the doppler shifted electron modes,

$$(\omega - k_y v_0)^2 \sim \frac{\omega_{pe}^2 k_z^2}{k_y^2 (\omega_{pe}^2 / \omega_{ce}^2) + k^2}$$

ANALYTICAL INSTABILITY ANALYSIS

COLD PLASMA LIMIT

Modified Two Stream Instability

Electron Cyclotron Drift Instability –

This is a 1D instability that happens when $k_{\|}/k_{\perp} \ll 1$,

$$k_y^2\left(1-\left(\frac{\omega_{pe}^2}{{\omega_e}^2-\omega_{ce}^2}\right)-\left(\frac{\omega_{pi}^2}{{\omega_i}^2}\right)\right)$$
$$-k_y^4\left[r_{Le}^2\left(\frac{3\omega_{pe}^2}{({\omega_e}^2-4\omega_{ce}^2)({\omega_e}^2-\omega_{ce}^2)}\right)+3r_{Li}^2\frac{\omega_{pi}^2}{{\omega_i}^4}\right]=0$$

ANALYTICAL INSTABILITY ANALYSIS

COLD PLASMA LIMIT

Modified Two Stream Instability

Electron Cyclotron Drift Instability –

To 2nd order we have,

$$k_y^2\left(1-\frac{\omega_{pe}^2}{\left(\omega-k_y v_{e0}\right)^2-\omega_{ce}^2}-\frac{\omega_{pi}^2}{\omega^2}\right)=0$$

This may be thought of as coupling between $\omega^2=\omega_{pe}^2$, and the doppler shifted coupling of electrons,

$$(\omega-kv_0)^2=\omega_{pe}^2+\omega_{ce}^2$$

FUTURE PROSPECTS

- Numerical Stimulation and matching of the above analytical insights.
- Extending the Analysis to Hot / Semi-Hot Plasma Limits.
- Fiddling with the initial Maxwellian Velocity distribution.
- Higher order Approximations to include highly non linear behaviour.

- **CAN THESE EXPLAIN THE ANOMOLOUS ELECTRON TRANSPORT PROBLEM?**

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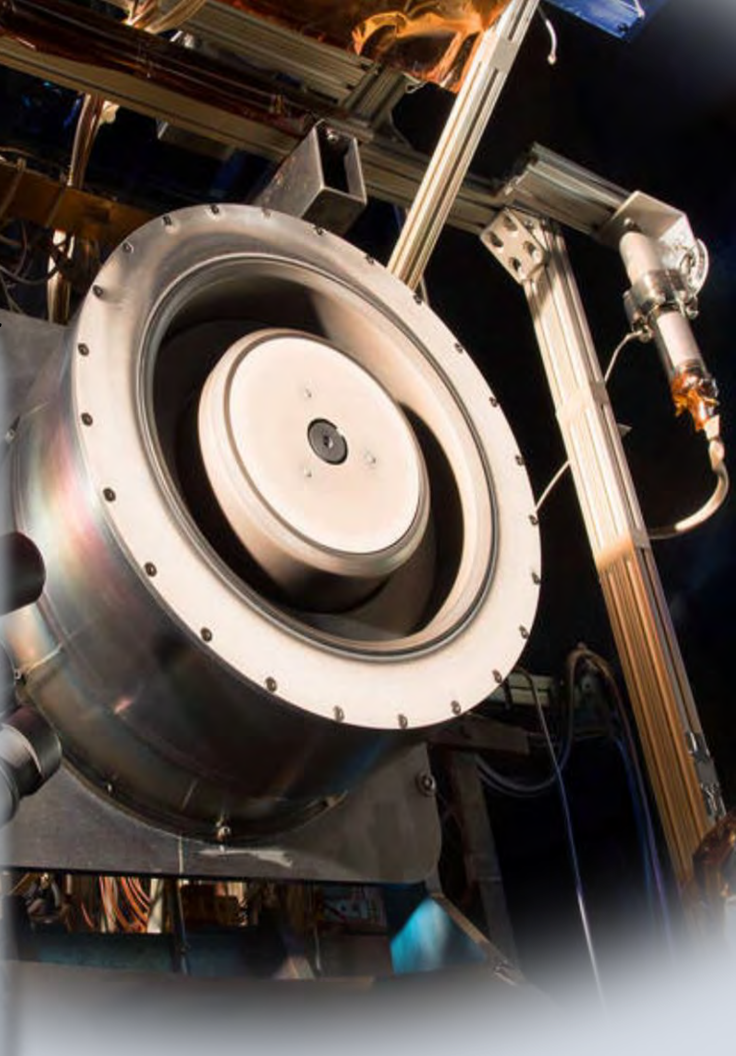

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THANK YOU

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Plasma Instabilities in Hall Thrusters,

SEM 8 Project Presentation

THE END